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Journal of Geometry and Physics 49 (2004) 1–20

JOURNAL OF GEOMETRY AND PHYSICS

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Stress–energy–momentum tensors for natural constrained variational problems

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Received 28 February 2003

Abstract

Under certain parameterization conditions for the "infinitesimal admissible variations", we propose a theory for constrained variational problems on arbitrary bundles, which allows us to introduce in a very general way the concept of multi-momentum map associated to the infinitesimal symmetries of the problem. For natural problems with natural parameterization, a stress–energy–momentum tensor is constructed for each "admissible section" from the multi-momentum map associated to the natural lifting of vector fields on the base manifold. This tensor satisfies the typical properties of a stress–energy–momentum tensor (Diff(X)-covariance, Belinfante–Rosenfeld type formulas, etc.), and also satisfies corresponding conservation and Hilbert type formulas for natural problems depending on a metric. The theory is illustrated with several examples of geometrical and physical interest.

MSC: 58E30; 76M30; 83C40

Keywords: Natural Lagrangian densities; Poincaré–Cartan forms; Multi-momentum maps; Stress–energy–momentum tensors

1. Introduction

In [6] the authors generalized to higher order the method introduced by Gotay and Marsden [11] to construct stress–energy–momentum tensors for first order variational problems from the multi-momentum map associated to the natural lifting of vector fields of the base manifold by infinitesimal symmetries of the problem. More precisely, the main result in [6] is the following:

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0393-0440/$ – see front matter
doi:10.1016/S0393-0440(03)00058-5

Given a Lagrangian density $\mathcal{L}\omega$ on the bundle J^kY of k-jets of sections of a natural bundle $p : Y \to X$ with differential index 1 ($\mathcal{L} \in C^\infty(J^kY)$, $\omega =$ volume element on X), for each section $s \in \Gamma(X, Y)$ there exists a unique tensor $\mathcal{T}(s) \in \Gamma(X, T^*X \otimes \Lambda^{n-1}T^*X)$ such that

$$i_D\mathcal{T}(s) = \mu_\Theta(s)(D) + \mathrm{d}\alpha, \quad D \in \mathfrak{X}(X),$$

where $\mu_\Theta : \Gamma(X, Y) \to \mathfrak{X}(X)^* \otimes \Gamma(X, \Lambda^{n-1}T^*X)$ is the multi-momentum map associated to the infinitesimal symmetries $\mathfrak{X}(X)$ and to any Poincaré–Cartan form, Θ, of the variational problem ($\alpha = (n-2)$-form on X depending on Θ, s and D).

The tensor so constructed satisfies the typical properties of a stress–energy–momentum tensor (Diff(X)-covariance, Belinfante–Rosenfeld type formulas, etc.), and admits also a Hilbert type explicit expression as well as the corresponding conservation law for those problems depending on a metric.

In the spirit of the program of Lagrangian reduction in its most recent formulations [2–4,7,12], according to which a certain kind of variational problems, called "reducible" ones, can be "reduced" to lower order constrained variational problems, it seems appropriate to study the concept of stress–energy–momentum tensor with regard to such a reduction procedure. Electromagnetic field theory ("field intensities" versus "potentials") and relativistic fluids (Eulerian and Lagrangian pictures) are two typical examples of this situation. The subject is still more relevant, taking into account the lack at the present time of a reasonable definition of Poincaré–Cartan form for constrained problems, a concept which, as we have seen, represents the foundation of the notion of stress–energy–momentum tensor in the ordinary case.

Motivated by this question, in the present work we tackle the study of the concept of stress–energy–momentum tensor for natural constrained variational problems as a first stage in the program of reduction we have mentioned.

The outline of the paper is as follows. In Section 2 we propose a formulation for the constrained variational calculus where the "admissible infinitesimal variations" $\mathcal{A}_S$ of the problem are considered as one more datum, on equal footing with the Lagrangian density $\mathcal{L}\omega$ and the constraint S. If we now impose the existence of a vector bundle $q : E \to Y$ (bundle of parameters) such that for each admissible section $s \in \Gamma_S(X, Y)$, the vertical components along s of $\mathcal{A}_S$ are the image of sections of s^*E by a certain first order differential operator P_s from that bundle to the bundle s^*VY (parameterization condition 2.2), it is now possible to obtain a first variation formula for constrained problems (Theorem 2.4), from which we have the way clear to develop in this new situation the variational theory. In particular, taking into consideration the "boundary term" of this formula, it is possible to state a Noether theory for infinitesimal symmetries of constrained problems and to give a definition of the corresponding multi-momentum map.

Section 3 is dedicated to natural problems. With an obvious definition of this concept in the constrained case (Definition 3.1), if we admit that the parameterization of the problem is also natural (condition 3.2), it is then possible to introduce the concept of multi-momentum map associated to the infinitesimal symmetries defined by the vector fields of the base manifold (Definition 3.5). In these conditions the main result of this section is Theorem 3.6, which introduces the concept of stress–energy–momentum tensor for constrained problems. In Section 4, natural problems depending on a metric are considered, obtaining an interesting generalization of the typical Hilbert formula and conservation law for this kind of tensor (Theorem 4.2 and Corollaries 4.3 and 4.4).

The theory we present is illustrated with four examples of geometrical and physical interest: Euler–Poincaré equations of principal connections (Example 1), and H-minimal Lagrangian submanifolds (Example 2), considered at the end of Section 2, and electromagnetism (Example 3) and relativistic fluids (Example 4), treated in Section 5.

Regarding the notations and other preliminary aspects, these are the same as in [6], of which this paper can be considered as a natural continuation.

2. Calculus of variations with constraints

Our starting point will be a Lagrangian density $\mathcal{L}\omega$ on the bundle $j^k p : J^k Y \to X$ of the k-jets of local sections of a fiber bundle $p : Y \to X$ on an n-dimensional oriented manifold ($\mathcal{L} \in \mathcal{C}^\infty(J^k Y)$ and ω a volume element on X), a submanifold $S \subseteq J^k Y$ such that $(j^k p)(S) = X$ (the constraint), and a subalgebra $\mathcal{A}_S$ of the Lie algebra $\mathfrak{X}^{(k)}(Y)$ of infinitesimal contact transformations of order k, tangential to the submanifold S (the variation algebra). On the subset $\Gamma_S(X, Y) = \{s \in \Gamma(X, Y) \mid \operatorname{Im} j^k s \subseteq S\}$ of sections that satisfy the constraint one has the functional:

$$\mathbb{L}(s) = \int_X (j^k s)^* \mathcal{L}\omega$$

defined for sections $s \in \Gamma_S(X, Y)$ for which the previous integral exists.

If $\mathcal{A}_S^c$ is the subalgebra of those elements in the variation algebra $\mathcal{A}_S$ whose support projects onto a compact subset of X, we may define the differential of the functional $\mathbb{L}$ at any section $s \in \Gamma_S(X, Y)$ by the rule:

$$(\delta_s \mathbb{L})(D) = \int_X (j^k s)^* L_D(\mathcal{L}\omega) \in \mathbb{R}, \quad D \in \mathcal{A}_S^c. \tag{2.1}$$

From here the definition of a critical section can be given as follows.

Definition 2.1. A section $s \in \Gamma(X, Y)$ is critical for the constrained variational problem of Lagrangian density $\mathcal{L}\omega$, constraint submanifold $S \subseteq J^k Y$ and variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}^{(k)}(Y)$ if s satisfies the constraint, i.e., $s \in \Gamma_S(X, Y)$, and the differential $\delta_s \mathbb{L} : \mathcal{A}_S^c \to \mathbb{R}$ at the section s vanishes.

From now on, we shall assume the variation algebra $\mathcal{A}_S$ to satisfy the following condition.

Condition 2.2 (parameterization condition). There exists a vector bundle $q : E \to Y$ (bundle of parameters) and a vector bundle morphism $P : J^1(E/X)_{J^1Y} \to (VY)_{J^1Y}$ (where $J^1(E/X)_{J^1Y}$ is the vector bundle $j^1 q : J^1(E/X) \to J^1 Y$ and where VY_{J^1Y} is the pull-back of VY to $J^1 Y$) such that for each admissible section $s \in \Gamma_S(X, Y)$ the first order differential operator $P_s : \Gamma(X, s^* E) \to \Gamma(X, s^* VY)$ defined by $P_s(e_s) = P(j^1 e_s)$ (parameterization operator) satisfies

$$P_s(\Gamma(X, s^*E)) = \mathcal{A}_s^{\mathrm{v}} = \{D_s^{\mathrm{v}} | D \in \mathcal{A}_S\}, \qquad P_s(\Gamma^{\mathrm{c}}(X, s^*E)) = \mathcal{A}_s^{\mathrm{c}} = \{D_s^{\mathrm{v}} | D \in \mathcal{A}_S^{\mathrm{c}}\},$$

where $D_s^{\mathrm{v}} = \theta^1(D)_{j^k s}$ denotes the vertical component along s of the vector field D.

Given a local fibered coordinate system (x^ν, y^j, q^i) for $q : E \to Y$ (where (x^ν, y^j) is a local fibered coordinate system for $p : Y \to X$ and the functions q^i are linear on the fibers of $q : E \to Y$), the vector bundle morphism $P : J^1(E/X) \to VY_{J^1Y}$ can be expressed, with respect to local coordinates (x^ν, y^j_β) and $(x^\nu, y^j_\beta, q^i_\alpha)$ $0 \le |\alpha|, |\beta| \le 1$ induced on the respective 1-jet bundles, as

$$P(x^\nu, y^j_\beta, q^i_\alpha) = (P_i^{j\alpha}(x^\nu, y^j_\beta) q^i_\alpha) \left(\frac{\partial}{\partial y^j}\right).$$

Thus, if a section s is defined by $y^j = y^j(x)$, the corresponding differential operator P_s can be expressed as

$$P_s(q^i(x)) = \left(P_i^j\left(x^\nu, \frac{\partial y^j(x)}{\partial x^\beta}\right) q^i(x) + P_i^{j\mu}\left(x^\nu, \frac{\partial y^j(x)}{\partial x^\beta}\right)\frac{\partial q^i(x)}{\partial x^\mu}\right)\frac{\partial}{\partial y^j}. \tag{2.2}$$

Proposition 2.3 (definition of the operator P_s^+ adjoint to P_s). *There exists a unique first order differential operator* $P_s^+ : \Gamma(X, s^*VY^* \otimes \Lambda^n T^*X) \to \Gamma(X, s^*E^* \otimes \Lambda^n T^*X)$ *such that*

$$\langle P_s(e), \mathcal{E}\rangle = \langle e, P_s^+(\mathcal{E})\rangle + \mathrm{d}(\langle \sigma P_s(e), \mathcal{E}\rangle) \tag{2.3}$$

$e \in \Gamma(X, s^*E)$, $\mathcal{E} \in \Gamma(X, s^*V^*Y \otimes \Lambda^n T^*X)$, *where* σP_s *is the symbol of the operator* P_s *and where the bilinear products are the obvious ones*.

Proof. Let $\{e_1, \dots .e_{m'}\}$ be a local basis of E associated to the local fibered coordinate system (x^ν, y^j, q^i) chosen for E. Let $\{\omega^1, \dots, \omega^{m'}\}$ denote its dual basis. Following (2.2), for any sections $e = q^i(x)e_i \in \Gamma(X, s^*E)$ and $\mathcal{E} = g_j(x)\,\mathrm{d}y^j \otimes \omega \in \Gamma(X, s^*VY^* \otimes \Lambda^n T^*X)$, we have

$$\begin{aligned}\langle P_s(q^i e_i), g_j\,\mathrm{d}y^j \otimes \omega\rangle &= \left([P_s]_i^j q^i g_j + [P_s]_i^{j\mu}\frac{\partial q^i}{\partial x^\mu} g_j\right)\omega \\ &= \left([P_s]_i^j q^i g_j + \frac{\partial}{\partial x^\mu}([P_s]_i^{j\mu} q^i g_j) - q^i\frac{\partial}{\partial x^\mu}([P_s]_i^{j\mu} g_j)\right)\omega \\ &= q^i\left([P_s]_i^j g_j - \frac{\partial}{\partial x^\mu}([P_s]_i^{j\mu} g_j)\right)\omega + \mathrm{d}([P_s]_i^{j\mu} q^i g_j \omega_\mu) \\ &= \langle q^i e_i, P_s^+(g_j\,\mathrm{d}y^j \otimes \omega)\rangle + \mathrm{d}(\langle \sigma P_s(q^i e_i), g_j\,\mathrm{d}y^j \otimes \omega\rangle),\end{aligned} \tag{2.4}$$

where we denote $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, $\omega_\mu = i_{\partial/\partial x^\mu}\omega$, and $[P_s]_i^j(x) = P_i^j(x^\nu, (\partial y^j(x)/\partial x^\beta))$, $[P_s]_i^{j\mu}(x) = P_i^{j\mu}(x^\nu, (\partial y^j(x)/\partial x^\beta))$ for any section $s \in \Gamma_S(X, Y)$ with equations $y^j = y^j(x)$.

As can be seen, the term $[P_s]_i^{j\mu} q^i g_j \omega_\mu$ is obtained by contraction of $\mathcal{E}$ with $\sigma P_s(q^i e_i) \in \Gamma(X, T^*X \otimes s^*VY)$, σP_s being the symbol of the differential operator P_s. The decomposition given is therefore global and independent of the chosen coordinate system, and the (globally well-defined) difference between $\langle P_s(e), \mathcal{E}\rangle$ and $\mathrm{d}(\langle \sigma P_s(e), \mathcal{E}\rangle)$ is a tensor on the component $e \in \Gamma(X, s^*E)$.

To prove uniqueness, let us suppose that there are two such decompositions: $\langle P_s(e), \mathcal{E}\rangle = \langle e, P_s^+(\mathcal{E})\rangle + \mathrm{d}\langle \sigma P_s(e), \mathcal{E}\rangle = \langle e, P_s^\star(\mathcal{E})\rangle + \mathrm{d}\langle \overline{\sigma P_s}(e), \mathcal{E}\rangle$. We would have $((P_s^+ - P_s^\star)(\mathcal{E}))(e) = d \circ (\langle \overline{\sigma P_s} - \sigma P_s, \mathcal{E}\rangle)(e)$, a decomposition of the form $T(e) = d \circ H(e)$, for a morphism of vector bundles T.

We shall then prove that there is no morphism T of a vector bundle s^*E to the bundle of n-forms that factors through the exterior derivative d and a differential operator H (apart from $T = 0$). Indeed, if $T(e) = \mathrm{d}(H(e)) \in \Gamma(X, \Lambda^n T^*X)$ is not 0 at some point $p \in X$, there would exist a function $f \in \mathcal{C}^\infty(X)$ with compact support in a neighborhood of p such that $\int_X f \cdot T(e) \neq 0$, but then

$$0 \neq \int_X f \cdot T(e) = \int_X T(f \cdot e) = \int_{\mathrm{Supp}(f)} \mathrm{d}(H(f \cdot e)) = \int_{\partial\, \mathrm{Supp}(f)} H(f \cdot e) = 0,$$

where the last equality holds because $\mathrm{Supp}(H(f \cdot e)) \subseteq \mathrm{Supp}(f)$ for any differential operator H. The only possibility is, then, $T = 0$.

Moreover, in this case $H(e)$ is a closed $(n-1)$-form for each $e \in \Gamma(X, s^*E)$, hence if H is $\mathcal{C}^\infty(X)$-linear, $\mathrm{d}(H(f \cdot e)) = \mathrm{d}(f \cdot H(e)) = \mathrm{d}f \wedge H(e) = 0\ \forall f \in \mathcal{C}^\infty(X)$ and therefore $H(e) \in \Gamma(X, \Lambda^{n-1} T^*X)$ vanishes for any $e \in \Gamma(X, s^*E)$. Both components of our decomposition (2.3) are unique. □

Formula (2.3) provides a commutation rule which, when applied to the first variation formula of the calculus of variations without constraints [6, Theorem 2.5], leads to the following fundamental result.

Theorem 2.4 (constrained first variation formula). *For any admissible section $s \in \Gamma_S(X, Y)$ and any admissible infinitesimal variation $D \in \mathcal{A}_S$ of a constrained variational problem with Lagrangian density $\mathcal{L}\omega$ on J^kY, constraint submanifold $S \subseteq J^kY$ and variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}^{(k)}(Y)$, satisfying the parameterization condition* 2.2, *one has*

$$(j^k s)^* L_D(\mathcal{L}\omega) = \langle e_{D_s^{\mathrm{v}}}, P_s^+ \mathcal{E}(s)\rangle + \mathrm{d}[(j^{2k-1}s)^* i_{D_{(2k-1)}} \Theta + \langle \sigma P_s(e_{D_s^{\mathrm{v}}}), \mathcal{E}(s)\rangle], \quad (2.5)$$

where $\mathcal{E}(s)$ and Θ are, respectively, the Euler–Lagrange operator and any Poincaré–Cartan form for the Lagrangian density $\mathcal{L}\omega$ as a problem without constraints and where $e_{D_s^{\mathrm{v}}} \in \Gamma(X, s^*E)$ is any section such that $P_s(e_{D_s^{\mathrm{v}}}) = D_s^{\mathrm{v}}$.

The linear functional $\delta_s\mathbb{L}$ defined by (2.1) *will be given by the formula*:

$$(\delta_s\mathbb{L})(D) = \int_X \langle e_{D_s^{\mathrm{v}}}, P_s^+ \mathcal{E}(s)\rangle, \quad D \in \mathcal{A}_S^{\mathrm{c}}, \quad (2.6)$$

where $e_{D_s^{\mathrm{v}}} \in \Gamma^{\mathrm{c}}(X, s^*E)$ is any section such that $P_s(e_{D_s^{\mathrm{v}}}) = D_s^{\mathrm{v}}$.

Proof. Parameterization condition 2.2 of the variation algebra $\mathcal{A}_S$ allows the substitution of $\theta^1(D)_{j^k s}$ in the first variation formula (2.3) of [6] by $P_s(e_{D^{\rm v}_s})$ for some section $e_{D^{\rm v}_s} \in \Gamma(X, s^*E)$. Formula (2.5) is now obtained by applying Proposition 2.3.

Formula (2.6) can now be obtained in a direct way taking $D \in \mathcal{A}^{\rm c}_S$. □

From formula (2.6), taking into account the arbitrariness of the section $e_{D^{\rm v}_s} \in \Gamma^{\rm c}(X, s^*E)$, we obtain the following corollary.

Corollary 2.5. *A section $s \in \Gamma(X, Y)$ is critical for the constrained variational problem if and only if*

$$\operatorname{Im} j^k s \subseteq S, \qquad P^+_s \mathcal{E}(s) = 0. \tag{2.7}$$

The first group of equations $\operatorname{Im} j^k s \subseteq S$ are the constraints, k-order differential equations on the components of s, while the second group, following the explicit expressions in (2.4), is given by

$$\begin{aligned} 0 = P^+_s \mathcal{E}(s) = \Bigg\{ & P^j_i \left(x^\nu, \frac{\partial y^j(x)}{\partial x^\beta} \right) \cdot \mathcal{E}_j \left(x^\nu, \frac{\partial^{|\sigma|} y^j(x)}{\partial x^\sigma} \right) \\ & - \frac{\partial}{\partial x^\mu} \left(P^{j\mu}_i \left(x^\nu, \frac{\partial y^j(x)}{\partial x^\beta} \right) \cdot \mathcal{E}_j \left(x^\nu, \frac{\partial^{|\sigma|} y^j(x)}{\partial x^\sigma} \right) \right) \Bigg\} \omega^i \otimes \omega. \end{aligned} \tag{2.8}$$

The mapping $P^+\mathcal{E} : s \in \Gamma_S(X, Y) \mapsto P^+_s \mathcal{E}(s) \in \Gamma(X, E \otimes \Lambda^n T^* X)$ is determined when we fix the vector bundle of parameters E and morphism P of parameterization for $\mathcal{A}_S$. We shall call it the Euler–Lagrange operator of the constrained variational problem parameterized by P.

In this framework, all the typical questions of the calculus of variations without constraints (infinitesimal symmetries and Noether theorems, second variation, Hamiltonian formalism, etc.) can be developed in a similar way. In particular, Noether theory can be established as follows.

Definition 2.6. An infinitesimal symmetry of a constrained variational problem with Lagrangian density $\mathcal{L}\omega$ on $J^k Y$, constraint submanifold $S \subseteq J^k Y$ and variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}^{(k)}(Y)$, is a vector field $D \in \mathcal{A}_S$ such that $L_D(\mathcal{L}\omega) = 0$.

Theorem 2.7 (Noether). *Given a constrained variational problem verifying parameterization condition* 2.2, *if $D \in \mathcal{A}_S$ is an infinitesimal symmetry and s is a critical section of the problem, then*

$$\mathrm{d}[(j^{2k-1}s)^* i_{D_{(2k-1)}} \Theta + \langle \sigma P_s(e_{D^{\rm v}_s}), \mathcal{E}(s) \rangle] = 0, \tag{2.9}$$

where Θ is any Poincaré–Cartan form of the Lagrangian density $\mathcal{L}\omega$ as a problem without constraints and $e_{D^{\rm v}_s} \in \Gamma(X, s^*E)$ is any section such that $P_s(e_{D^{\rm v}_s}) = D^{\rm v}_s$.

Proof. It suffices to apply the formula of variation (2.5), taking into account that, since $D \in \mathcal{A}_S$ is an infinitesimal symmetry of the problem, $L_D(\mathcal{L}\omega) = 0$, and since $s \in \Gamma_S(X, Y)$ is critical, $P^+_s \mathcal{E}(s) = 0$. □

Following the same route as in the calculus of variations without constraints, this framework allows us to define for any subalgebra $\mathcal{D}_S$ of the Lie algebra of infinitesimal symmetries of a constrained variational problem, a multi-momentum map $\mu : \Gamma_S(X, Y) \to \mathcal{D}_S^* \otimes \Lambda^{n-1}T^*X$ by the rule:

$$[\mu(s)](D) = [(j^{2k-1}s)^* i_{D_{(2k-1)}}\Theta + \langle \sigma P_s(e_{D_S^{\vee}}), \mathcal{E}(s)\rangle], \quad s \in \Gamma_S(X, Y), \quad D \in \mathcal{D}_S. \tag{2.10}$$

Remark 2.8. Variational problems without constraints correspond to the case $S = J^kY$, $\mathcal{A}_S = \mathfrak{X}^{(k)}(Y)$, $E = VY$ and $P = \mathrm{Id}$, so that for any section $s \in \Gamma(X, Y)$ there holds $P_s = \mathrm{Id}$, $P_s^+ = \mathrm{Id}$ and $\sigma P_s = 0$, so formulas (2.5)–(2.9) transform into the corresponding ones for problems without constraints.

Remark 2.9. The opposite case in complexity arises when the parameterization of the problem is given by differential operators P_s of arbitrary order **a** which depend on the section s up to a certain order **b**, i.e., the bundle of parameters is a vector bundle $q : E \to J^bY$, the morphism P is a vector bundle morphism $P : J^a(E/X)_{J^{a+b}Y} \to (VY)_{J^{a+b}Y}$ and the parameterization operators $P_s : \Gamma(X, (j^bs)^*E) \to \Gamma(X, s^*VY)$, $s \in \Gamma_S(X, Y)$ are given by the formula $P_s(e_{j^bs}) = P(j^ae)$. In this case, an analogue to Proposition 2.3 can be given, the adjoint of the operators P_s exist and are univocally defined, the term σP_s is replaced by a certain $(a-1)$-order differential operator, and from here all the results above can be recovered in the same way as has been explained. For more details on this generalization, the reader is referred to [1,16].

We will finish this section with two examples that illustrate this approach in a very clear way.

Example 1. Euler–Poincaré equations for principal connections.

Let $p : P \to X$ be a principal bundle with structural group G and $\mathrm{Ad}\,P$ the corresponding adjoint bundle. Let $\mathcal{C}(P) = J^1P/G$ be the affine bundle of connections on P [8–10], modelled over the vector bundle $T^*X \otimes \mathrm{Ad}\,P$. On this bundle we shall consider the constrained variational problem with Lagrangian density $\mathcal{L}\omega$ ($\mathcal{L} \in C^\infty(\mathcal{C}(P))$), constraint submanifold $S = \{j_x^1\gamma \mid (\mathrm{Curv}\,\gamma)_x = 0\}$ and variation algebra, $\mathcal{A}_S$, the natural representation over the bundle of connections of the Lie algebra $\mathrm{aut}\,P$ of infinitesimal automorphisms of the principal bundle P.

The parameterization condition 2.2 for this constrained problem holds taking as bundle of parameters $E = (\mathrm{Ad}\,P)_{\mathcal{C}(P)}$, the pull-back to $\mathcal{C}(P)$ of the adjoint bundle of P, and the vector bundle morphism $P : (j_x^1B, j_x^1\gamma) \in J^1(E/X) \mapsto ((\mathrm{d}^\gamma B)_x, j_x^1\gamma) \in V(\mathcal{C}(P))_{J^1\mathcal{C}(P)}$, where the differential $(\mathrm{d}^\gamma B)_x \in T_x^*X \otimes \mathrm{Ad}_x\,P$ can be seen as an element of $V(\mathcal{C}(P))$ via the natural identification of this bundle with $(T^*X \otimes \mathrm{Ad}\,P)_{\mathcal{C}(P)}$.

The parameterization operator $P_\gamma : \Gamma(X, \mathrm{Ad}\,P) \to \Gamma(X, T^*X \otimes \mathrm{Ad}\,P)$ is the differential d^γ with respect to the connection γ and its adjoint is the corresponding divergence operator div_γ. The characterization of critical sections of this problem by Corollary 2.5 is then given by Euler–Poincaré equations [7]:

$$\operatorname{Curv}\gamma = 0, \qquad \operatorname{div}_\gamma(\mathcal{E}_{\mathcal{L}\omega}(\gamma)) = 0,$$

where $\mathcal{E}_{\mathcal{L}\omega}$ is the Euler–Lagrange operator associated to the Lagrangian density $\mathcal{L}\omega$ as a problem without constraints.

Example 2. H-minimal Lagrangian submanifolds.

Given a symplectic manifold (M_{2n}, Ω_2) endowed with a Riemannian metric g, as is known, this theory deals with the study of Lagrangian submanifolds $X_n \subseteq M_{2n}$ (i.e., $\Omega_2|_{X_n} = 0$) that minimize the functional $X_n \mapsto g$-area of X_n, with respect to certain variations ("Hamiltonian" variations), that conserve the "Lagrangianity". Given one of these submanifolds, according to the Darboux–Weinstein Theorem, there exists a symplectic diffeomorphism of a tubular neighborhood of $X \subseteq M$ with a neighborhood of the zero section in T^*X, which inherits the Riemannian structure given on M. Taking into account that a section $\eta : X \to T^*X$ is Lagrangian if and only if $\mathrm{d}\eta = 0$, the problem transforms into a constrained variational problem with constraint submanifold $S = \{j^1_x\eta \mid (\mathrm{d}\eta)_x = 0\}$ and variation algebra $\mathcal{A}_S = \mathrm{d}\mathcal{C}^\infty(X) \oplus \widetilde{\mathfrak{X}(X)}$ given by the vertical vector fields defined by exact 1-forms on X and by the natural lifting of vector fields from X to T^*X. The Lagrangian density in this example is the functional $\mathcal{L}\omega : \eta \mapsto g$-area element of η.

The parameterization condition 2.2 holds again taking as bundle of parameters the trivial bundle $E = T^*X \times \mathbb{R}$ and the morphism $P : (j^1_x\eta, j^1_x f) \in J^1(E/X) \mapsto (j^1_x\eta, (\mathrm{d}f)_x) \in V(T^*X)_{J^1T^*X}$ defined by the exterior derivative via the natural identification $V(T^*X) = T^*X \times T^*X$.

The adjoint operator (making the obvious identifications using the metric tensor g) for the exterior derivative d is the codifferential $\delta = *\,\mathrm{d}*$, where $*$ is the Hodge operator of the manifold X with respect to its Riemannian metric tensor. The characterization of critical sections given in Corollary 2.5 is then

$$\mathrm{d}\eta = 0, \qquad \delta H_\eta = 0,$$

where H_η is the polar 1-form with respect to the given Riemannian metric g of the mean curvature vector along the submanifold η.

3. Natural constrained variational problems

In the following, $p : Y \to X$ will be a natural bundle [5,6,14,15]. For each $\varphi \in \operatorname{Diff}(X)$ let $\tilde{\varphi} \in \operatorname{Diff}(Y)$ be the natural lifting of φ to Y and for $D \in \mathfrak{X}(X)$ let $\tilde{D} \in \mathfrak{X}(Y)$ be the natural lifting of D to Y.

Definition 3.1. A constrained variational problem with Lagrangian density $\mathcal{L}\omega$ on J^kY, constraint submanifold $S \subseteq J^kY$ and variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}^{(k)}(Y)$ is natural if:

(1) The bundle $p : Y \to X$ is natural.

(2) The natural liftings $\tilde{D}_{(k)} \in \mathfrak{X}^{(k)}(Y)$ of vector fields $D \in \mathfrak{X}(X)$ are infinitesimal symmetries of the constrained variational problem, i.e., $\tilde{D}_{(k)} \in \mathcal{A}_S$ and $L_{\tilde{D}_{(k)}}(\mathcal{L}\omega) = 0$.

Moreover, we shall assume that these variational problems satisfy the following refinement of condition 2.2 of parameterization.

Condition 3.2 (natural parameterization condition).

(1) The bundle of parameters $q : E \to Y$ is natural (considered as a bundle over X), so that for each $\varphi \in \mathrm{Diff}(X)$, the natural lifting $\tilde{\varphi}_E$ on E is q-projected onto the natural lifting $\tilde{\varphi}$ on Y.
(2) The vector bundle morphism $P : J^1(E/X)_{J^1Y} \to VY_{J^1Y}$ is natural and hence so are the parameterization operators $P_s : \Gamma(X, s^*E) \to \Gamma(X, s^*VY)$, $s \in \Gamma(X, Y)$, i.e.:

$$P_{\varphi\cdot s}(\varphi \cdot e) = \varphi \cdot P_s(e) \quad \forall\, e \in \Gamma(X, s^*E),$$

where $\varphi \cdot s = \tilde{\varphi} \circ s \circ \varphi^{-1}$, $\varphi \cdot (P_s(e)) = \varphi_* \circ (P_s(e)) \circ \varphi^{-1}$ and $\varphi \cdot e = \tilde{\varphi}_E \circ e \circ \varphi^{-1}$.
(3) The induced morphism $D \in \mathfrak{X}(X) \mapsto (\tilde{D}_{(k)})^{\mathrm{v}}_s \in \Gamma(X, s^*VY)$ factors through P_s by natural first order differential operators $J_s : \mathfrak{X}(X) \to \Gamma(X, s^*E)$, i.e:

$$P_s \circ J_s(D) = (\tilde{D}_{(k)})^{\mathrm{v}}_s \quad \forall\, s \in \Gamma_S(X, Y),$$
$$J_{\varphi\cdot s}(\varphi \cdot D) = \varphi \cdot J_s(D) \quad \forall\, D \in \mathfrak{X}(X)$$

Remark 3.3. In the case of natural variational problems without constraints, $E = VY$, $P = \mathrm{Id}$ and J_s is the natural lifting:

$$\begin{array}{rccc} J_s : & \mathfrak{X}(X) & \to & \Gamma(X, s^*VY) \\ & D & \mapsto & \tilde{D}^{\mathrm{v}}_s \end{array}$$

If J_s is a first order differential operator, we have described in [6] how to define a stress–energy–momentum tensor using the symbol of the operators J_s.

As a first result towards this objective for the constrained case, we have the following proposition.

Proposition 3.4. *The Euler–Lagrange operator* $P^+\mathcal{E} : s \in \Gamma_S(X, Y) \mapsto P^+_s\mathcal{E}(s) \in \Gamma(X, s^*E^* \otimes \Lambda^n T^*X)$ *of a natural constrained variational problem satisfying the natural parameterization condition* 3.2 *is* $\mathrm{Diff}(X)$*-covariant, i.e.*:

$$P^+\mathcal{E}(\varphi \cdot s) = \varphi \cdot P^+\mathcal{E}(s), \tag{3.1}$$

where $\varphi\cdot$ *stands for the natural operation of* $\mathrm{Diff}(X)$ *on the different objects where it is applied.*

Proof. For the covariance of Euler–Lagrange operator $\mathcal{E}(s)$ of $\mathcal{L}\omega$ as a problem without constraints, see [13,16]:

$$\mathcal{E}(\varphi \cdot s) = \varphi \cdot \mathcal{E}(s). \tag{3.2}$$

Now in the case of constrained problems, from the naturalness of the differential operator P:

$$\varphi \cdot (P_s(e)) = P_{\varphi\cdot s}(\varphi \cdot e)$$

and from the definition of the adjoint operators P_s^+ and $P_{\varphi\cdot s}^+$ we get

$$\begin{aligned} \langle P_{\varphi\cdot s}(\varphi \cdot e), \mathcal{E}_{\varphi\cdot s}\rangle &= \langle \varphi \cdot e, P_{\varphi\cdot s}^+(\mathcal{E}_{\varphi\cdot s})\rangle + \mathrm{d}(\langle \sigma P_{\varphi\cdot s}(\varphi \cdot e), \mathcal{E}_{\varphi\cdot s}\rangle), \\ \langle P_{\varphi\cdot s}(\varphi \cdot e), \mathcal{E}_{\varphi\cdot s}\rangle &= \langle \varphi \cdot P_s(e), \mathcal{E}_{\varphi\cdot s}\rangle = \varphi \cdot \langle P_s(e), \varphi^{-1} \cdot \mathcal{E}_{\varphi\cdot s}\rangle \\ &= \varphi \cdot \langle e, P_s^+(\varphi^{-1} \cdot \mathcal{E}_{\varphi\cdot s})\rangle + \mathrm{d}(\varphi \cdot \langle \sigma P_s(e), \varphi^{-1} \cdot \mathcal{E}_{\varphi\cdot s}\rangle) \\ &= \langle \varphi \cdot e, (\varphi \cdot P_s^+)(\mathcal{E}_{\varphi\cdot s})\rangle + \mathrm{d}(\varphi \cdot \langle \sigma P_s(e), \varphi^{-1} \cdot \mathcal{E}_{\varphi\cdot s}\rangle) \end{aligned}$$

for any section $\mathcal{E}_{\varphi\cdot s} \in \Gamma(X, (\varphi \cdot s)^* VY^* \otimes \Lambda^n T^* X)$, and for the natural definitions of $\varphi\cdot$ on the different objects. By uniqueness of the adjoint operator, we conclude that

$$\varphi \cdot P_s^+ = P_{\varphi\cdot s}^+, \qquad \varphi \cdot \sigma P_s = \sigma P_{\varphi\cdot s}. \tag{3.3}$$

Taking both equalities (3.2) and (3.3) we get (3.1), proving the proposition. □

On the other hand, consideration of the first order differential operators J_s, $s \in \Gamma_S(X, Y)$ from point (3) in the natural parameterization condition 3.2 allows for this kind of variational problems to define a multi-momentum map associated to the Lie algebra $\mathfrak{X}(X)$ of vector fields on X, by the following definition.

Definition 3.5. We shall call the multi-momentum map of the problem the map $\mu : \Gamma_S(X, Y) \to \mathfrak{X}(X)^* \otimes \Lambda^{n-1} T^* X$ given by

$$\mu(s)(D) = (j^{2k-1}s)^* i_{\tilde{D}_{(2k-1)}} \Theta + \langle \sigma P_s(J_s(D)), \mathcal{E}(s)\rangle, \quad D \in \mathfrak{X}(X), \tag{3.4}$$

where Θ is a Poincaré–Cartan form of the Lagrangian density $\mathcal{L}\omega$ as a variational problem without constraints.

From this concept, which depends on the chosen Poincaré–Cartan form, we are in a situation to prove the main result of this section.

Theorem 3.6 (main theorem). *Given a natural constrained variational problem, there holds*:

(1) *For each section $s \in \Gamma_S(X, Y)$ there exists a unique tensor $\mathcal{T}(s) : \mathfrak{X}(X) \to \Gamma(X, \Lambda^{n-1} T^* X)$ such that, for any multi-momentum map μ associated to the problem and any vector field $D \in \mathfrak{X}(X)$, there holds*

$$i_D \mathcal{T}(s) = \mu(s)(D) + \mathrm{d}\alpha, \tag{3.5}$$

where α is a $(n-2)$-form on X depending on Θ, (P, J_s) and D.

(2) *The tensor $\mathcal{T}(s)$ is explicitly given by*

$$i_D \mathcal{T}(s) = -(P^+ \mathcal{E}(s))(\sigma J_s(D)), \tag{3.6}$$

where $P^+\mathcal{E}$ is the Euler–Lagrange operator of the constrained variational problem, σJ_s is the symbol of the operator J_s which defines the natural parameterization, and the contractions are the obvious ones.

(3) *The assignment $s \in \Gamma_S(X, Y) \mapsto \mathcal{T}(s) \in \Gamma(X, T^*X \otimes \Lambda^{n-1}T^*X)$ is* Diff(X)*-covariant, i.e., for every diffeomorphism $\varphi \in \mathrm{Diff}(X)$ one has*

$$\mathcal{T}(\varphi \cdot s) = \varphi \cdot (\mathcal{T}(s))$$

where $\varphi \cdot s = \tilde{\varphi} \circ s \circ \varphi^{-1}$, $(\varphi \cdot (\mathcal{T}(s)))(\varphi_* D) = (\varphi^{-1})^*(\mathcal{T}(s)(D))$, $\tilde{\varphi} : Y \to Y$ being the natural lifting of φ on the bundle $p : Y \to X$.

Proof. Considering the first variation formula (2.5) and the definition (3.4) of the multi-momentum map μ for symmetries $\tilde{D}$, we get

$$\begin{aligned} 0 = (j^k s)^* L_{\tilde{D}_{(k)}}(\mathcal{L}\omega) &= \langle J_s(D), P^+\mathcal{E}(s)\rangle + \mathrm{d}(\mu(s)(D)) \\ &= \langle D, J_s^+ \circ P_s^+\mathcal{E}(s)\rangle + \mathrm{d}(\mu(s)(D) + \langle \sigma J_s(D), P^+\mathcal{E}(s)\rangle). \end{aligned}$$

But, as we know from the proof of Proposition 2.3, there are no non-trivial morphisms of vector bundles from TX to $\Lambda^n T^*X$ that factor through the exterior derivative, hence:

$$J_s^+ \circ P_s^+\mathcal{E}(s) = 0, \qquad \mathrm{d}(\mu(s)(D) + \langle \sigma J_s(D), P^+\mathcal{E}(s)\rangle) = 0. \tag{3.7}$$

The differential operator $\mu(s) + (P^+\mathcal{E}(s))(\sigma J_s)$ takes values on closed $(n-1)$-forms. The $\Lambda^{n-1}T^*X$-valued 1-covariant tensor $-(P^+\mathcal{E}(s))(\sigma J_s)$ is independent on the choice of Θ and differs from the multi-momentum map in a closed $(n-1)$-form. Applying the same general principle as in [6, p. 52], the difference is an exact $(n-1)$-form. We thus obtain the proof of (3.5) and (3.6)

For the uniqueness, as we saw in the proof of Proposition 2.3, there are no vector bundle morphisms between TX and $\Lambda^{n-1}T^*X$ that take values on closed $(n-1)$-forms except for the trivial one. Thus, if $\mathcal{T}(s)$ and $\mathcal{T}'(s)$ satisfy (3.5) for some choice of the multi-momentum map (depending on the chosen Poincaré–Cartan form), $\mathcal{T}(s) = \mathcal{T}'(s)$. The explicit formula (3.6) will ensure the independence of this uniqueness from the chosen μ.

Following Proposition 3.4, we have $P^+\mathcal{E}(\varphi \cdot s) = \varphi \cdot P^+\mathcal{E}(s)$ and the naturalness assumption on J produces $\sigma J_{\varphi \cdot s} = \varphi \cdot \sigma J_s$. The combination of both expressions yields the Diff(X)-covariance. □

Eq. (3.5) characterizing the stress–energy–momentum tensor $\mathcal{T}(s)$ can be interpreted as a generalization to natural constrained variational problems of the "Belinfante–Rosenfeld formula", from which this tensor is obtained by adding to the value $\mu(s) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{X}(X), \Gamma(X, T^*X \otimes \Lambda^{n-1}T^*X))$ of the multi-momentum map (which is not a tensor) a "corrective term" given by the last summand of that formula. On the other hand, (3.6) constitutes the basic formula that will allow us to generalize in the constrained variational calculus the classic Hilbert expression $\mathcal{T}(s) = 2\delta\mathcal{L}/\delta g$ of the stress–energy–momentum tensor to the case that the natural problem has a metric as parameter.

4. Natural constrained variational problems depending on a metric

Recalling the case of problems without constraints [6, Section 4], let $\mathcal{L}\omega$ be a natural Lagrangian density on the k-jet bundle $J^k(\mathcal{M} \times_X Y)$ of the natural fibered product $\pi \times p : \mathcal{M} \times_X Y \to X$, where $\pi : \mathcal{M} \to Y$ is the bundle of non-singular metrics of given signature on X and where $p : Y \to X$ is a natural bundle. If $\omega_{\mathcal{M}}$ is the horizontal volume element on $\mathcal{M} \times_X Y$ given by $(\omega_{\mathcal{M}})_{(g_x, y_x)} = (\pi \times p)^* \omega_{g_x}$ (ω_g = volume element associated to the metric tensor g), then the Lagrangian density $\mathcal{L}\omega$ can be expressed in the form $L\omega_{\mathcal{M}}$, where $L \in \mathcal{C}^\infty(J^k(\mathcal{M} \times_X Y))$ is an invariant function for the natural action of $\mathrm{Diff}(X)$ on $J^k(\mathcal{M} \times_X Y)$. In these conditions for each metric g the natural immersion of bundles on X, $i_g : Y \hookrightarrow \mathcal{M} \times_X Y$, defines a Lagrangian density on $J^k Y$ by the rule: $L_g \omega_g = (j^k i_g)^*(L\omega_{\mathcal{M}})$, thus obtaining a family $\{L_g \omega_g\}$ of Lagrangian densities on $J^k Y$ parameterized by metrics $g \in \Gamma(X, \mathcal{M})$ on X. Following this approach, seeking our objective, how can we introduce the "constraints", "parameterization conditions", "naturalness", etc. in the variational problem $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M} \times_X Y)$ and in the family $\{L_g \omega_g\}$ on $J^k Y$?

First of all, we shall consider a constraint submanifold $S \subseteq J^k(\mathcal{M} \times_X Y)$ and a variation algebra $\mathcal{A}_S \subseteq \mathfrak{X}^{(k)}(\mathcal{M} \times_X Y)$ for the variational problem with Lagrangian density $\mathcal{L}\omega_{\mathcal{M}}$ on $J^k(\mathcal{M} \times_X Y)$ without any restriction on the metric component of the problem in a sense which will be made clear below.

On the other hand, the constrained variational problem we shall consider will be a natural one (Definition 3.1) endowed with a natural parameterization (P, J) (condition 3.2) whose bundle of parameters is $(S^2 T^* X \oplus E)_{\mathcal{M} \times_X Y}$ (E a natural vector bundle on Y) and where the parameterization operator $P_{(g,s)}$ along any section $(g, s) \in \Gamma_S(X, \mathcal{M} \times_X Y)$ has the form

$$\begin{array}{llll} P_{(g,s)}: & \Gamma(X, S^2 T^* X \oplus (g, s)^* E) & \to & \Gamma(X, S^2 T^* X \oplus s^* VY) \\ & (S_2, e) & \mapsto & (S_2, P^Y_{(g,s)}(S_2, e)) \end{array}$$

Under these conditions, for any metric g, the submanifold $S_g = (j^k i_g)^{-1} S \subseteq J^k Y$ and the algebra

$$\mathcal{A}_{S_g} = \{D_Y \in \mathfrak{X}^{(k)}(Y) \mid (j^k i_g)_*(D_Y) = D|_{\mathrm{Im}(j^k i_g)} \quad \text{for some } \; D \in \mathcal{A}_S\}$$

define a constrained variational problem for the Lagrangian density $L_g \omega_g$ on $J^k Y$.

As additional hypothesis in this framework, we shall assume that, for any admissible section $(g, s) \in \Gamma_S(X, \mathcal{M} \times_X Y)$, the differential operators $P^Y_{(g,s)}|_{0 \oplus s^* E}$ define a parameterization for the subspaces of infinitesimal admissible variations $(\mathcal{A}_{S_g})^{\mathrm{v}}_s$ and $(\mathcal{A}_{S_g})^{\mathrm{c}}_s$ for the latter variational problem.

From this point, proceeding as in the case without constraints [6, Definition 4.1] we may give the following definition.

Definition 4.1. The stress–energy–momentum tensor of the constrained variational problem with Lagrangian density $L_g \omega_g$ on $J^k Y$, constraint submanifold $S_g \subseteq J^k Y$ and variation algebra $\mathcal{A}_{S_g} \subseteq \mathfrak{X}^{(k)}(Y)$, is the correspondence that assigns to each admissible section $s \in \Gamma_{S_g}(X, Y)$ the tensor $\mathcal{T}_g(s) = \mathcal{T}(g, s)$, where $\mathcal{T}$ is the stress–energy–momentum tensor corresponding to the section $(g, s) \in \Gamma_S(X, \mathcal{M} \times_X Y)$ of the natural constrained variational problem with Lagrangian density $L\omega_{\mathcal{M}}$ on $J^k(\mathcal{M}\times_X Y)$, constraint submanifold $S\subseteq J^k(\mathcal{M}\times_X Y)$ and variation algebra $\mathcal{A}_S\subseteq\mathfrak{X}^{(k)}(\mathcal{M}\times_X Y)$.

Using the volume element ω_g the tensor $\mathcal{T}_g(s)\in\Gamma(X,T^*X\otimes\Lambda^{n-1}T^*X)$ can be seen as a 1-covariant, 1-contravariant tensor $(T_g)^1_1(s)\in\Gamma(X,TX\otimes T^*X)$ defined by $i_{(T_g)^1_1(s)(K)}\omega_g=\mathcal{T}_g(s)(K)$ or, lowering or raising an index by means of g, we get the tensors $(T_g)_2(s)\in\Gamma(X,T^*X\otimes T^*X)$ or $(T_g)^2(s)\in\Gamma(X,TX\otimes TX)$. We shall use the different versions of $(\mathcal{T}_g)(s)$ as we need them.

The tensor we have just defined satisfies two new properties (Hilbert's formula and the divergence formula) whose precise statement is given below.

As is well known [6, (3.12)], for any vector field $K\in\mathfrak{X}(X)$, its natural lifting $\tilde{K}^{\mathcal{M}}$ on the bundle $\mathcal{M}$ has a vertical component $(\tilde{K}^{\mathcal{M}})^{\mathrm{v}}_g\in\Gamma(X,g^*V\mathcal{M})=\Gamma(X,S^2T^*X)$ along a section $g\in\Gamma(X,\mathcal{M})$ given by the expression:

$$(\tilde{K}^{\mathcal{M}})^{\mathrm{v}}_g=-L_Kg=-2\,\mathrm{Sym}(\mathrm{d}^{\nabla_g}i_Kg),$$

where Sym is the symmetrization operator and d^{∇_g} is the covariant derivative with respect to the Levi–Civita connection ∇_g associated to the metric g. Thus, as $P_{(g,s)}\circ J_{(g,s)}(K)=((\tilde{K}^{\mathcal{M}})^{\mathrm{v}}_g,(\tilde{K}^{Y})^{\mathrm{v}}_s)=(-2\,\mathrm{Sym}(\mathrm{d}^{\nabla_g}i_Kg),(\tilde{K}^{Y})^{\mathrm{v}}_s)$ the differential operators $J_{(g,s)}$ must be of the form

$$J_{(g,s)}(K)=(-2\,\mathrm{Sym}(\mathrm{d}^{\nabla_g}i_Kg),J^E_{(g,s)}(K))\in\Gamma(X,S^2T^*X\oplus(g,s)^*E),\quad K\in\mathfrak{X}(X).\tag{4.1}$$

On the other hand, let $P^{S^2}_{(g,s)}:S_2\in\Gamma(X,S^2T^*X)\mapsto P^Y_{(g,s)}(S_2,0)\in\Gamma(X,s^*VY)$ and $P^E_{(g,s)}:e\in\Gamma(X,(g,s)^*E)\mapsto P^Y_{(g,s)}(0,e)\in\Gamma(X,s^*VY)$ be the differential operators induced by the parameterization operator $P^Y_{(g,s)}$ and $\mathcal{E}_{\mathcal{M}}(g,s)\in\Gamma(X,S^2TX\otimes\Lambda^nT^*X)$ and $\mathcal{E}_Y(g,s)\in\Gamma(X,(s^*VY)^*\otimes\Lambda^nT^*X)$ the two components given by the decomposition $(g,s)^*V(\mathcal{M}\times_X Y)=g^*V\mathcal{M}\oplus s^*VY$ of the Euler–Lagrange operator associated to $L\omega_{\mathcal{M}}$ as a problem without constraints. In this situation, there holds the following theorem.

Theorem 4.2. *The stress–energy–momentum tensor $(\mathcal{T}_g)(s)$ is given by*

$$(\mathcal{T}_g)(s)=2(\mathcal{E}_{\mathcal{M}}+(P^{S^2})^+\mathcal{E}_Y)^1_1(g,s)-((P^E)^+\mathcal{E}_Y)(g,s)\cdot\sigma J^E(g,s),\tag{4.2}$$

where for S^2TX-valued n-forms $\mathcal{E}$, the expression $\mathcal{E}^1_1$ represents the corresponding T^*X-valued $(n-1)$-form defined by $\mathcal{E}^1_1(K)=c^1_1(i_{g(K)}\mathcal{E})$.

Proof. Formula (3.6) for the natural problem $(\pi\times p,L\omega_{\mathcal{M}},S,\mathcal{A}_S)$ leads to

$$\mathcal{T}_g(s)=\mathcal{T}(g,s)=-P^+_{(g,s)}(\mathcal{E}_{\mathcal{M}}(g,s),\mathcal{E}_Y(g,s))\cdot\sigma J_{(g,s)},\tag{4.3}$$

where the terms can be easily calculated:

$$\begin{aligned}P^+_{(g,s)}(\mathcal{E}_{\mathcal{M}}(g,s),\mathcal{E}_Y(g,s))&=(\mathcal{E}_{\mathcal{M}}(g,s),0)+(P^Y_{(g,s)})^+\mathcal{E}_Y(g,s)\\&=(\mathcal{E}_{\mathcal{M}}+(P^{S^2})^+\mathcal{E}_Y,(P^E)^+\mathcal{E}_Y)(g,s),\end{aligned}$$

whose first component is a symmetric tensor. The symbol of $J_{(g,s)}$, following (4.1) is

$$\sigma J_{(g,s)}(K,\theta) = [-2\,\mathrm{Sym}(i_K g \otimes \theta), \sigma J^E_{(g,s)}(K,\theta)], \quad K \in \mathfrak{X}(X), \ \ \theta \in \mathfrak{X}(X)^*.$$

Substituting these expressions in (4.3) produce

$$\mathcal{T}_g(s) = 2(\mathcal{E}_{\mathcal{M}}(g,s) + (P^{S^2}_{(g,s)})^+ \mathcal{E}_Y(g,s))^1_1 - (P^E_{(g,s)})^+ \mathcal{E}_Y(g,s) \cdot \sigma J^E_{(g,s)},$$

thus proving the theorem. □

Due to the latter hypothesis of our setting, the second component $(P^E)^+\mathcal{E}_Y$ in the formula is the Euler–Lagrange operator of the constrained problem $(p, L_g\omega_g, S_g, \mathcal{A}^Y_{S_g})$. Thus, we obtain the following corollary.

Corollary 4.3 (Hilbert's formula). *If $s \in \Gamma_{S_g}(X,Y)$ is a critical section of the constrained variational problem $(p, L_g\omega_g, S_g, \mathcal{A}_{S_g})$ or if the natural lifting $K \mapsto J^E_{(g,s)}(K)$ has differential order* 0, *there holds*

$$(T_g)_2(s) = 2\left[\frac{\delta L\omega_{\mathcal{M}}}{\delta g} + (P^{S^2}_{(g,s)})^+ \frac{\delta L\omega_{\mathcal{M}}}{\delta y}\right]. \tag{4.4}$$

Proof. It follows from (4.2), where the component $((P^E_{(g,s)})^+\mathcal{E}_Y(g,s)) \cdot \sigma J^E_{(g,s)}$ vanishes if $s \in \Gamma_{S_g}(X,Y)$ is critical (i.e., $(P^E_{(g,s)})^+\mathcal{E}_Y(g,s) = 0$) or if $J^E_{(g,s)}$ has differential order 0 (i.e., $\sigma J^E_{(g,s)} = 0$). □

Corollary 4.4 (divergence formula).

$$\begin{aligned}\mathrm{div}_g((T_g)_2(s)) \otimes \omega_g &= -(J^E_{(g,s)})^+((P^E)^+\mathcal{E}_Y(g,s)) \\ &\quad - \mathrm{div}_g(((P^E)^+\mathcal{E}_Y)(g,s) \cdot \sigma J^E(g,s))_2 \otimes \omega_g.\end{aligned} \tag{4.5}$$

In particular, for any critical section $s \in \Gamma_{S_g}(X,Y)$ of $(p, L_g\omega_g, S_g, \mathcal{A}_{S_g})$, there holds

$$\mathrm{div}_g(T_g)_2(s) = 0$$

Proof. The divergence of any 2-covariant tensor $\mathcal{E}_2 \in \Gamma(X, S^2T^*X)$ can be given in terms of the differential operator $K \in \mathfrak{X}(X) \mapsto 2\,\mathrm{Sym}(\mathrm{d}^{\nabla_g} i_K g) \in \Gamma(X, S^2T^*X)$ as in [6, 4.8], obtaining for any $\mathcal{E} = (\mathcal{E}_{\mathcal{M}}, \mathcal{E}_Y) \otimes \omega_g \in \Gamma(X, (S^2TX \oplus s^*V^*Y) \otimes \Lambda^n T^*X)$ the expression:

$$(J_{(g,s)})^+(\mathcal{E}) = \mathrm{div}_g(\mathcal{E}_{\mathcal{M}})_2 \otimes \omega_g + (J^E_{(g,s)})^+(\mathcal{E}_Y \otimes \omega_g).$$

For the Euler–Lagrange operator $\mathcal{E} = (2(\mathcal{E}_{\mathcal{M}} + (P^{S^2})^+\mathcal{E}_Y)^2 \otimes \omega_{\mathcal{M}}, (P^E)^+\mathcal{E}_Y)$ we know by (3.7) that $J^+\mathcal{E}(g,s) = 0$, so

$$\mathrm{div}_g\, 2(\mathcal{E}_{\mathcal{M}} + (P^{S^2})^+\mathcal{E}_Y)_2(g,s) \otimes \omega_g = -(J^E_{(g,s)})^+((P^E)^+\mathcal{E}_Y)(g,s).$$

Hence, the divergence of the stress–energy–momentum tensor given by formula (4.2) is

$$\begin{aligned}
&\mathrm{div}_g((T_g)_2(s))\otimes\omega_g\\
&\quad=\mathrm{div}_g\,2(\mathcal{E}_{\mathcal{M}}+(P^{S_2})^+\mathcal{E}_Y)_2(g,s)\otimes\omega_g-\mathrm{div}_g(((P^E)^+\mathcal{E}_Y)\cdot\sigma J^E)_2(g,s)\otimes\omega_g\\
&\quad=-(J^E_{(g,s)})^+((P^E)^+\mathcal{E}_Y(g,s))-\mathrm{div}_g(((P^E)^+\mathcal{E}_Y)(g,s)\cdot\sigma J^E(g,s))_2\otimes\omega_g,
\end{aligned}$$

thus proving the formula. □

The divergence formula (4.5) gives a necessary condition $\mathrm{div}_g(T_g)_2(s)=0$ for a section $s\in\Gamma_{S_g}(X,Y)$ to be critical $((P^E)^+\mathcal{E}_Y(g,s)=0)$. Depending on the differential operators J^E, this could also be a sufficient condition (see the example of the relativistic fluid).

5. Examples

Examples 1 and 2 which we have studied in Section 2 have been cases of non-natural problems with a mainly geometric interest. In this section we are going to illustrate the theory with two typical natural problems arising from physics: the electromagnetic field and the relativistic fluids on space–time with a Lorentz metric as parameter.

Example 3 (electromagnetism). In [6] this theory was developed as a natural variational problem without constraints on $J^1(\mathcal{M}\times_{X_4}T^*X_4)$, where $\pi:\mathcal{M}\to X_4$ is the bundle of Lorentz metrics on a four-dimensional manifold X_4 and $p:T^*X_4\to X_4$ is the bundle of "electromagnetic potentials", A, describing the electromagnetic field by $F=\mathrm{d}A$. In this section we will tackle this problem in terms of the electromagnetic field F without making use of the potentials.

Let us consider on the bundle $\pi\times p:\mathcal{M}\times_{X_4}\Lambda^2T^*X_4\to X_4$ the 0-order Lagrangian density given by $\mathcal{L}(g,F)\omega_{\mathcal{M}}=L_g(F)\omega_g=(1/2)\|F\|_g^2\omega_g$. Consider the first order constraint on $(g,F)\in\Gamma(X_4,\mathcal{M}\times_{X_4}\Lambda^2T^*X_4)$ given by the submanifold $S=\{j^1_x(g,F)\in J^1(\mathcal{M}\times_{X_4}\Lambda^2T^*X_4)|(\mathrm{d}f)_x=0\}$. Admissible sections are 2-forms F on X_4 which satisfy the first group of Maxwell's equations $\mathrm{d}F=0$. In order to fix the variation algebra $\mathcal{A}_S$ on $J^1(\mathcal{M}\times_{X_4}\Lambda^2T^*X_4)$ tangential to S, we consider, through the natural identification $V_{(g_x,F_x)}(\mathcal{M}\times_{X_4}\Lambda^2T^*X_4)=S^2T^*_xX_4\oplus\Lambda^2T^*_xX_4$, vertical vector fields D_{S_2} defined by sections $S_2\in\Gamma(X_4,S^2T^*X_4)$, vertical vector fields $D_{\mathrm{d}\bar\eta}$ defined by closed 2-forms $\mathrm{d}\bar\eta\in\mathrm{d}\Omega^1_c$ that are exterior derivatives of 1-forms $\bar\eta$ with compact support, and vector fields $\tilde K$ given by the natural lifting of vector fields $\mathfrak{X}(X_4)$ to the bundle $\mathcal{M}\times_{X_4}\Lambda^2T^*X_4$. The variation algebra $\mathcal{A}_S$ is going to be the 1-jet extension to $J^1(\mathcal{M}\times_{X_4}\Lambda^2T^*X_4)$ of the above defined algebra $S^2T^*X_4\oplus\mathrm{d}\Omega^1_c\oplus\widetilde{\mathfrak{X}(X_4)}$. Its infinitesimal admissible variations are given by

$$D^{\mathrm{v}}_{(g,F)}=(S_2-L_Kg,\mathrm{d}(\bar\eta-i_KF))$$

for any $D=D_{S_2}+D_{\mathrm{d}\bar\eta}+\tilde K\in\Gamma(X_4,S^2T^*X\oplus\mathrm{d}\Omega^1_c\oplus\widetilde{\mathfrak{X}(X_4)})$.

The natural parameterization of this natural constrained variational problem is given by

$$\begin{array}{llll} P: & J^1(S^2T^*X_4 \oplus T^*X_4) & \to & V(\mathcal{M} \times_{X_4} \Lambda^2T^*X_4) \\ & j^1_x(S_2, A')_{(g,F)} & \mapsto & (S_2(x), (\mathrm{d}A')_x)_{(g_x,F_x)} \end{array}$$

$$\begin{array}{llll} J_{(g,F)}: & \mathfrak{X}(X_4) & \to & \Gamma(X_4, S_2T^*X_4 \oplus T^*X_4) \\ & K & \mapsto & (-L_K g, -i_K F) \end{array}$$

Fixing the metric g, the constrained variational problem defined on $p: \Lambda^2T^*X_4 \to X_4$ will be given by Lagrangian density $L_g(F)\omega_g = (1/2)\|F\|^2_g\omega_g$, constraint submanifold $S_g = \{j^1_x F \mid (\mathrm{d}f)_x = 0\}$, variation algebra $\mathcal{A}_{S_g} = \mathrm{d}\Omega^1_c \oplus \widetilde{\mathfrak{X}(X_4)}$, and parameterization operator $P^E_{(s,g)}: A' \in \Gamma(X_4, T^*X_4) \mapsto \mathrm{d}A' \in \Gamma(X_4, \Lambda^2T^*X_4)$.

In this case adjunction formula (2.3) can be expressed as

$$\langle \mathrm{d}A', \mathcal{E}\rangle_g\omega_g = \langle A', \delta_g\mathcal{E}\rangle_g\omega_g + \mathrm{d}(*_g\mathcal{E} \wedge A') \quad \forall \mathcal{E} \in \Gamma(X_4, \Lambda^2T^*X_4),$$

where $*_g$ is Hodge's operator defined by $\langle \eta_2, \mathcal{E}\rangle_g\omega_g = *_g\mathcal{E} \wedge \eta_2$ and $\delta_g = *_g^{-1} \circ \mathrm{d} \circ *_g$.

The linear component in A' gives the expression for the adjoint operator $(P^E)^+_F : \mathcal{E} \in \Gamma(X_4, \Lambda^2T^*X_4) \mapsto \delta_g\mathcal{E} \in \Gamma(X_4, T^*X_4)$ and the second component gives the morphism $(\sigma P^E)_F : (A', \mathcal{E}) \in \Gamma(X_4, T^*X \oplus \Lambda^2T^*X_4) \mapsto *_g\mathcal{E} \wedge A' \in \Gamma(X_4, \Lambda^3T^*X_4)$.

If $\mathcal{E}_g(F) \otimes \omega_g$ is the value of the Euler–Lagrange operator at some admissible section $F \in \Gamma_{S_g}(X_4, \Lambda^2T^*X_4)$ as a problem without constraints ($\mathcal{E}_g(F) \in \Gamma(X_4, \Lambda^2T^*X_4)$ and duality given by the scalar product of 2-forms w.r.t. g), the corresponding first variation formula (2.5) for the constrained problem will be given by

$$F^*L_D(L_g\omega_g) = \langle A', \delta_g\mathcal{E}_g(F)\rangle_g\omega_g + \mathrm{d}[F^*i_D(L_g\omega_g) + *_g\mathcal{E}_g(F) \wedge A'],$$

where $A' = \bar{\eta} - i_K F$ is the 1-form that parameterizes the vertical component $D^{\mathrm{v}}_F = \mathrm{d}\bar{\eta} - L_K F$ along F of the variation $D = D_{\mathrm{d}\bar{\eta}} + \tilde{K} \in \mathcal{A}_{S_g}$.

For $L_g\omega_g(F) = (1/2)\|F\|^2_g\omega_g$, the Euler–Lagrange operator is $\mathcal{E}_g(F) \otimes \omega_g = F \otimes \omega_g$ and the first variation formula will be

$$F^*L_D(L_g\omega_g) = \langle A', \delta_g F\rangle_g\omega_g + \mathrm{d}[\tfrac{1}{2}\|F\|^2_g i_K\omega_g + *_g F \wedge A'],$$

where $D = D_{\mathrm{d}\bar{\eta}} + \tilde{K}$ and $A' = \bar{\eta} - i_K F$.

The Euler–Lagrange operator $P^+\mathcal{E}$ for the variational problem with constraints $(p, L_g\omega_g, S_g, \mathcal{A}_{S_g})$ is hence given by

$$P^+\mathcal{E}(F) = \delta_g F,$$

where the duality with $F^*V(\Lambda^2T^*X_4)$ is given by $\langle , \rangle_g\omega_g$. Critical sections are characterized by

$$\mathrm{d}F = 0, \qquad \delta_g F = 0.$$

Coming back to the natural problem defined on $\pi \times p : \mathcal{M} \times_{X_4} T^*X_4 \to X_4$, as in this case $J^{T^*X_4}_{(g,F)} : K \in \mathfrak{X}(X_4) \mapsto -i_K F \in \Gamma(X_4, T^*X_4)$ has differential order 0 and $P^{S^2}_{(g,F)} = 0$, Hilbert's formula (4.4) gives the explicit expression of stress–energy–momentum tensor for this constrained variational problem:

$$(T_g)_2(F) = 2\frac{\delta L\omega_{\mathcal{M}}}{\delta g} = \frac{1}{2}\|F\|_g^2 \cdot g - F_1^1 \cdot F_2.$$

This tensor does not coincide with the tensor obtained in [6] for the electromagnetic field described in terms of electromagnetic potentials. That tensor $(T_g)_2(A) = (1/2)\|\mathrm{d}A\|_g^2 \cdot g - (\mathrm{d}A)_1^1 \cdot (\mathrm{d}A)_2 - A \otimes \delta_g(\mathrm{d}A)$ was not projectable to $\Lambda^2 T^* X^4$ but coincides with the one obtained here along critical sections. This difference is due to the fact that the multi-momentum map provided by the theory of electromagnetic potentials does not project to the bundle of electromagnetic intensities $\Lambda^2 T^* X_4$. The stress–energy–momentum tensors in each case are related to different multi-momentum maps that coincide up to a closed term.

Example 4 (relativistic fluids). As is well known a relativistic perfect fluid on a Lorentz manifold (X_4, g) is a divergence-free time-like vector field D on X_4 ($\mathrm{div}_g D = 0$ and $g(D, D) < 0$) which satisfies Euler's equation. Condition $\mathrm{div}_g D = 0$ can be interpreted as a constraint for a variational problem on the bundle $\mathcal{M} \times_{X_4} TX_4$, i.e., the constraint submanifold $\bar{S} = \{j_x^1(g, D) \mid (\mathrm{div}_g D)(x) = 0\} \subseteq J^1(\mathcal{M} \times_{X_4} TX_4)$.

Making use of the 1-jet extension $j^1\Phi$ of the bundle isomorphism $\Phi : (g_x, D_x) \in \mathcal{M} \times_{X_4} TX_4 \mapsto (g_x, i_{D_x}\omega_{g_x}) \in \mathcal{M} \times_{X_4} \Lambda^3 T^* X_4$, the submanifold $\bar{S}$ transforms into $S = j^1\Phi(\bar{S}) = \{j_x^1(g, \omega_3) \mid (\mathrm{d}\omega_3)(x) = 0\}$ which allows us to deal with the constraint by separating the metric and the fluid variables.

This suggests dealing with relativistic fluids as a constrained variational problem with Lagrangian density $\mathcal{L}\omega_{\mathcal{M}}$ on $J^1(\mathcal{M} \times_{X_4} \Lambda^3 T^* X_4)$, constraint submanifold $S = \{j_x^1(g, \omega_3) \mid (\mathrm{d}\omega_3)_x = 0\}$ and variation algebra $\mathcal{A}_S = S^2 T^* X_4 \oplus \widetilde{\mathfrak{X}(X_4)}$ defined in the same way as in the previous example.

The infinitesimal admissible variations are now

$$D^{\mathrm{v}}_{(g,\omega_3)} = (S_2 - L_K g, -\mathrm{d}i_K\omega_3)$$

for any $D = D_{S_2} + \tilde{K} \in \mathcal{A}_S$.

A natural parameterization can be given by

$$\begin{array}{llll} P: & J^1(S^2T^*X_4 \oplus TX_4)_{\mathcal{M}\times_{X_4}\Lambda^3T^*X_4} & \to & V(\mathcal{M} \times_{X_4} \Lambda^3 T^* X_4) \\ & j_x^1(S_2, K)_{(g,\omega_3)} & \mapsto & (S_2(x), -\mathrm{d}i_K\omega_3)_{(g,\omega_3)(x)} \end{array}$$

$$\begin{array}{llll} J_{(g,\omega_3)}: & \mathfrak{X}(X_4) & \to & \Gamma(X_4, S_2 T^* X_4 \oplus TX_4) \\ & K & \mapsto & (-L_K g, K) \end{array}$$

In this case, for a fixed metric g, the constrained variational problem on $p : \Lambda^3 TX_4 \to X_4$ will be given by a Lagrangian density $L_g\omega_g = (\widetilde{j^k i_g})^*(\mathcal{L}\omega_{\mathcal{M}})$, constraint submanifold $S_g = \{j_x^1\omega_3 \mid (\mathrm{d}\omega_3)_x = 0\}$, variation algebra $\mathcal{A}_{S_g} = \widetilde{\mathfrak{X}(X_4)}$ and parameterization operator:

$$P^E_{(g,\omega_3)} : K \in \Gamma(X_4, TX_4) \mapsto -\mathrm{d}i_K\omega_3 \in \Gamma(X_4, \Lambda^3 T^* X_4).$$

Then the adjunction formula (2.3) is analogous to the previous example:

$$\langle -\mathrm{d}i_K\omega_3, \mathcal{E}\rangle_g\omega_g = \langle i_K\omega_3, -\delta_g\mathcal{E}\rangle_g\omega_g + \mathrm{d}(*_g\mathcal{E}\wedge -i_K\omega_3) \quad \forall\,\mathcal{E}\in\Gamma(X_4, \Lambda^3T^*X_4)$$

and, by simple algebraic manipulations, can be expressed as

$$\langle -\mathrm{d}i_K\omega_3, \mathcal{E}\rangle_g\omega_g = \langle i_Kg, i_D\,\mathrm{d}*_g\mathcal{E}\rangle_g\omega_g + \mathrm{d}(*_g\mathcal{E}\wedge -i_K\omega_3),$$

where D is the field describing the fluid: $i_D\omega_g = \omega_3$.

The adjoint operator $(P^E_{(g,\omega_3)})^+$ and the morphism $\sigma P^E_{(g,\omega_3)}$ are given by

$$\begin{aligned}
&(P^E_{(g,\omega_3)})^+ : \mathcal{E}\in\Gamma(X_4, \Lambda^3T^*X_4) &&\mapsto\quad i_D\,\mathrm{d}*_g\mathcal{E}\in\Gamma(X_4, T^*X_4),\\
&\sigma P^E_{(g,\omega_3)} : (K,\mathcal{E})\in\Gamma(X_4, TX_4\oplus\Lambda^3T^*X_4) &&\mapsto\quad *_g\mathcal{E}\wedge -i_K\omega_3\in\Gamma(X_4, \Lambda^3T^*X_4).
\end{aligned}$$

If $\mathcal{E}_g(\omega_3)\otimes\omega_g$ is the value of the Euler–Lagrange operator at some admissible section $\omega_3\in\Gamma_{S_g}(X_4, \Lambda^3T^*X_4)$ as a problem without constraints ($\mathcal{E}_g(\omega_3)\in\Gamma(X_4, \Lambda^3T^*X_4)$ and duality given by the scalar product of 3-forms w.r.t. g), the corresponding first variation formula (2.5) for the constrained problem will be given by

$$\begin{aligned}
(j^k\omega_3)^*L_{\tilde{K}^{(k)}}(L_g\omega_g) &= i_K(i_D\,\mathrm{d}*_g\mathcal{E}_g(\omega_3))\omega_g\\
&\quad + \mathrm{d}[((j^{2k-1}\omega_3)^*i_{\tilde{K}^{(2k-1)}}\Theta - *_g\mathcal{E}_g(\omega_3)\wedge i_K\omega_3)].
\end{aligned}$$

Hence, the equations (2.7) that characterize critical sections are

$$\mathrm{d}\omega_3 = 0, \qquad i_D\,\mathrm{d}*_g\mathcal{E}_g(\omega_3) = 0.$$

In particular, perfect fluids are defined by 0-order natural Lagrangian densities $\mathcal{L}\omega_{\mathcal{M}}(g,\omega_3) = L_g\omega_g$ with $L_g = F(\rho)$ (where $\rho = \sqrt{-g(D,D)}$), so that the equations that characterize critical sections are

$$\mathrm{d}\omega_3 = 0, \qquad i_D\,\mathrm{d}\left(\frac{-F'(\rho)}{\rho}i_Dg\right) = 0,$$

which, interpreted via the isomorphism $\Phi : \mathcal{M}\times_{X_4}TX_4\to\mathcal{M}\times_{X_4}\Lambda^3T^*X_4$, expressing the Lagrangian as $L_g = -\rho(1+\epsilon(\rho))$ and defining $p = \rho^2(\mathrm{d}\epsilon(\rho)/\mathrm{d}\rho)$, $\mu = \rho(1+\epsilon(\rho))$ take the standard form of Euler equations:

$$\mathrm{div}_g\,\rho U = 0, \qquad U(p)\omega_U + \mathrm{d}p + (\mu+p)U^{\nabla_g}\omega_U = 0,$$

where $U = D/\rho$, $\omega_U = i_Ug$ and ∇_g is the Levi–Civita connection of g.

Coming back to the variational problem on $\mathcal{M}\times_{X_4}\Lambda^3T^*X_4$, using Hilbert's formula (4.4) we may give stress–energy–momentum tensors for these problems. As $J^{TX_4}_{(g,\omega_3)} = \mathrm{Id}$ has differential order 0 and $P^{S^2T^*X_4}_{(g,\omega_3)}(S_2) = 0$, we have

$$(T_g)_2(\omega_3) = 2\frac{\delta L\omega_{\mathcal{M}}}{\delta g}$$

for any admissible section $\omega_3\in\Gamma_{S_g}(X_4, \Lambda^3T^*X_4)$.

Using the divergence formula (4.5) in this case, where $J^{TX_4}_{(g,D)} = \mathrm{Id}$, the divergence of this tensor is

$$\mathrm{div}_g((T_g)^1_1(\omega_3)) = -i_D\,\mathrm{d}(*_g\mathcal{E}_g(\omega_3)),$$

thus, critical sections for the constrained variational problem $L_g\omega_g$ on $\Lambda^3T^*X_4$ are characterized by the constraint and the vanishing of its stress–energy–momentum tensor.

In the case of perfect fluids $L_g(\omega_3) = F(\rho) = -\rho(1+\epsilon(\rho))$, the stress–energy–momentum tensor and its divergence take the form:

$$(T_g)^2(D) = \mu U\otimes U + p(U\otimes U + g^{-1}),$$
$$\mathrm{div}_g((T_g)^1_1(D)) = U(p)\omega_U + \mathrm{d}p + (\mu+p)U^{\nabla_g}\omega_U$$

Remark 5.1. In this case the stress–energy–momentum tensor coincides with the one given in [6] for the variational problem without constraints given by the hydrodynamic potentials. This is due to the fact that the multi-momentum map in that case can be projected to the bundle $\mathcal{M}\times_{X_4}\Lambda^3T^*X_4$ as the multi-momentum map of the constrained problem.

Acknowledgements

This work has been partially supported by DGICYT (Spain), project number BFM2000-1314 and Junta de Castilla y León, project number SA085/01.

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